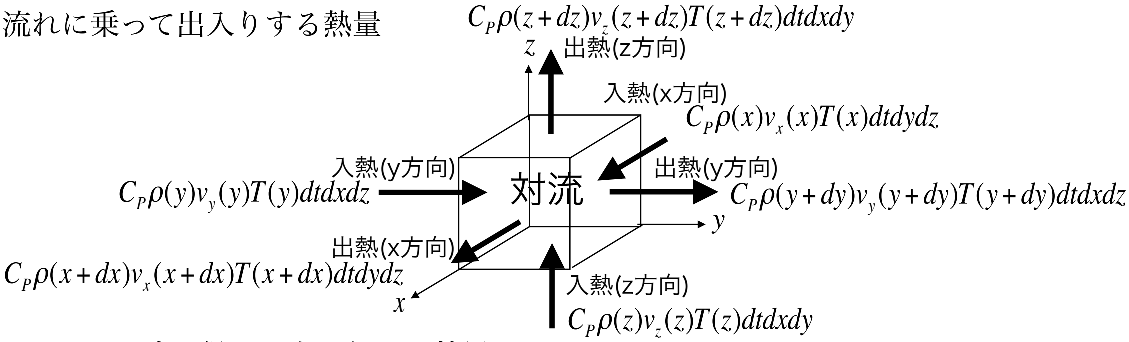
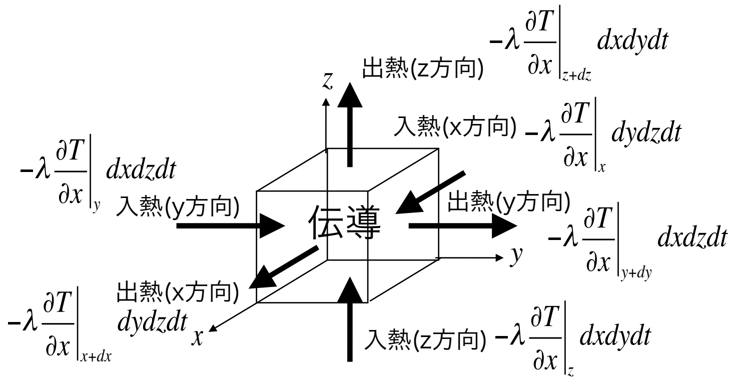
問題15　熱移動の式の導出

直角座標のCVに流れに乗って出入りする熱量



直角座標のCVにフーリエの式に従って出入りする熱量



直角座標のCVのdt時間での熱量収支

最初(t=t)のCVの熱量：$C_P\rho(t)T(t)dxdydz$

CVに入る熱量 − CVから出る熱量（移流分）：

$$+\ C_P\rho(x)v_x(x)T(x)dtdydz \ -\ C_P\rho(x+dx)v_x(x+dx)T(x+dx)dtdydz$$
$$+\ C_P\rho(y)v_y(y)T(y)dtdxdz \ -\ C_P\rho(y+dy)v_y(y+dy)T(y+dy)dtdxdz$$
$$+\ C_P\rho(z)v_z(z)T(z)dtdxdy \ -\ C_P\rho(z+dz)v_z(z+dz)T(z+dz)dtdxdy$$

速度式
フーリエの式の分

$$+\left(-\lambda\left.\frac{\partial T}{\partial x}\right|_{x} dydzdt\right) - \left(-\lambda\left.\frac{\partial T}{\partial x}\right|_{x+dx} dydzdt\right)$$
$$+\left(-\lambda\left.\frac{\partial T}{\partial x}\right|_{y} dxdzdt\right) - \left(-\lambda\left.\frac{\partial T}{\partial x}\right|_{y+dy} dxdzdt\right)$$
$$+\left(-\lambda\left.\frac{\partial T}{\partial x}\right|_{z} dxdydt\right) - \left(-\lambda\left.\frac{\partial T}{\partial x}\right|_{z+dz} dxdydt\right)$$

発熱分

$$+\ Qdxdydzdt$$

$$= C_P\rho(t+dt)T(t+dt)dxdydz$$

dt時間経過後(t=t+dt)のCVの熱量

Taylor展開あるいは微分の定義から

$$\rho(x+dx)v_x(x+dx)T(x+dx) = \rho(x)v_x(x)T(x) + \frac{\partial(\rho v_x T)}{\partial x}dx \qquad \left.\frac{\partial T}{\partial x}\right|_{x+dx} = \left.\frac{\partial T}{\partial x}\right|_{x} + \left.\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right)\right|_{x} dx \qquad \rho(t+dt)T(t+dt) = \rho(t)T(t) + \frac{\partial(\rho T)}{\partial t}dt$$

x,y,zそれぞれの方向に適用して整理する（移流分を移項して右辺と左辺を入れ替える

$$C_P\frac{\partial(\rho T)}{\partial t}dxdydzdt + C_P\left(\frac{\partial(\rho v_x T)}{\partial x} + \frac{\partial(\rho v_y T)}{\partial y} + \frac{\partial(\rho v_z T)}{\partial z}\right)dxdydzdt = -\lambda\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right)dxdydzdt + Qdxdydzdt$$

C_P(定数)で割って，非定常（時間微分）の項と移流項を展開する。

$$T\frac{\partial\rho}{\partial t} + \rho\frac{\partial T}{\partial t} + T\frac{\partial(\rho v_x)}{\partial x} + \rho v_x\frac{\partial T}{\partial x} + T\frac{\partial(\rho v_y)}{\partial y} + \rho v_y\frac{\partial T}{\partial y} + T\frac{\partial(\rho v_z)}{\partial z} + \rho v_z\frac{\partial T}{\partial z} = -\frac{\lambda}{C_P}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \frac{Q}{C_P}$$

$$T\left\{\frac{\partial\rho}{\partial t} + \frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z}\right\} + \rho\frac{\partial T}{\partial t} + \rho v_x\frac{\partial T}{\partial x} + \rho v_y\frac{\partial T}{\partial y} + \rho v_z\frac{\partial T}{\partial z} = -\frac{\lambda}{C_P}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \frac{Q}{C_P}$$

最初の項の｛ ｝内は連続の式よりゼロとなる。

$$\frac{\partial\rho}{\partial t} + \frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z} = 0$$

従って

$$\rho\frac{\partial T}{\partial t} + \rho v_x\frac{\partial T}{\partial x} + \rho v_y\frac{\partial T}{\partial y} + \rho v_z\frac{\partial T}{\partial z} = -\frac{\lambda}{C_P}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \frac{Q}{C_P}$$

さらにρで割って

$$\boxed{\frac{\partial T}{\partial t} + v_x\frac{\partial T}{\partial x} + v_y\frac{\partial T}{\partial y} + v_z\frac{\partial T}{\partial z} = -\nu\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \frac{Q}{\rho C_P}}$$